Numerical solution of the generalized Laplace equation by the boundary element method

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The boundary element method (BEM) has, in general, some advantages with respect to domain methods inasmuch as no internal discretization of the domain is required. This article shows that the generalized Laplace equation (GLE) can also be dealt with advantageously by BEM. The basic technique to achieve this consists of transforming the starting equation GLE into a constant-coefficient equation to which the standard BEM can be applied. The procedure is applied to solve numerically three test problems with known analytical solutions.

Keywords: mathematical models, generalized Laplace equation, boundary element method

Introduction

The boundary element method (BEM) has become a powerful and alternative method with respect to the well-known methods of finite differences (FDM) and finite elements (FEM). This is due mainly to its simplicity in the arrangement of the input data and to its applicability to problems defined on infinite domains. However, the application range of BEM is confined to simple differential equations, owing to the difficulty of finding a fundamental solution (FS) for a general differential operator. The applications of BEM mainly concern constant-coefficient differential equations, such as the Laplace, Poisson, Helmholtz, wave, and heat equations.[1–4]

It is well known that the Laplace equation, the first one to which BEM was applied, is a particular case of a more general equation, which we call the generalized Laplace equation (GLE).[5] Unfortunately it is not advantageous to apply BEM to the GLE, since its integral formulation requires the computation of both domain and boundary integrals.

Here we show that BEM can be successfully used to obtain an approximate numerical solution of the GLE. The procedure is not always applicable, at least from a mathematical point of view, but it gives sufficiently good results in many practical problems.

The technique consists of transforming the GLE into an equivalent differential equation in which the first partial derivatives have disappeared, thus rendering BEM applicable.

Unfortunately, the transformed equation also has nonconstant coefficients, so it would be difficult, in general, to find its FS, which, in any case, would be rather complicated.

However, under suitable conditions, the transformed equation can be approximated by a constant-coefficient equation, having a known FS, to which BEM can be applied in the usual fashion.

The numerical values of the original function can be easily obtained from the numerical solution of the approximated equation.

The whole procedure is presented for problems defined on two-dimensional domains; the extension to three-dimensional problems is straightforward.

To show how the technique works, three test problems, with known analytical solutions, are solved numerically.

The mathematical analysis of the computational error, related to the present technique, will be dealt with in a subsequent article.

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Transformation of the governing equation

Many physical problems are governed by the GLE[5]

$$\nabla \cdot [a(x, y)\nabla V] = 0 \tag{1}$$

where $\nabla \cdot$ and ∇ denote the divergence and gradient operators. Equation (1), for the unknown function $V = V(x, y)$, is supposed to be defined on a domain $D \subseteq R^2$ with boundary S. The function $a = a(x, y)$ is known and describes the physical properties of the domain D (i.e., it may be electrical, thermal, or hydraulic conductivity).

To define a unique solution V, we must introduce suitable boundary conditions, which *will* be dealt with below.

If function $a(x, y)$ is nowhere equal to zero on D, which usually happens in practice, we can write equation (1) in the form

$$\nabla^2 V + (1/a)\nabla V \cdot \nabla a = 0 \tag{2}$$

It is clear that for $a = \text{const.}$, equation (2) becomes the simpler Laplace equation.

Knowledge of a FS is not enough for applying the BEM to equation (2). Let G be a function satisfying

$$\nabla^2 G(P, Q) = \delta(P) - (1/a)\nabla G \cdot \nabla a \tag{3}$$

where P and Q are points in $D \cup S$, known as the observation point and the source point, respectively, and δ denotes the Dirac measure. Applying the third Green formula to the functions G and V, we have

$$\begin{aligned}\int_D (V\nabla^2 G - G\nabla^2 V)\, dD &= V(P) + \int_D \frac{1}{a}(G\nabla a \cdot \nabla V - V\nabla a \cdot \nabla G)\, dD \\ &= -\int_S \left(V\frac{\partial G}{\partial n} - G\frac{\partial V}{\partial n}\right) dS\end{aligned} \tag{4}$$

where n denotes the inward normal to S.

Equation (4) shows that an integral on D remains, with all related computational disadvantages (arrangement of an internal mesh, a larger algebraic system of equations to solve, etc.). This trouble is caused by the presence, in equation (2), of the first partial derivatives of the function V.

Let us see whether it is possible to transform equation (2) into an equivalent equation without first partial derivatives. It is impossible, as proved in Ref. 6, to transform equation (2) into a constant-coefficient equation.

Let us set

$$V(x, y) = u(x, y)H(x, y) \tag{5}$$

Substituting the required partial derivatives of V according to (5) in equation (2) and imposing the condition that the multiplier H render the coefficients of the terms $\partial u/\partial x$ and $\partial u/\partial y$ zero, we obtain an ordinary differential equation for H whose solution is

$$H(x, y) = [a(x, y)]^{-1/2} \tag{6}$$

With H defined by (6) and taking (5) into account, we obtain for equation (2)

$$\nabla^2 u + \left(\frac{|\nabla a|^2}{4a^2} - \frac{\nabla^2 a}{2a}\right)u = \nabla^2 u + f(x, y)u = 0 \tag{7}$$

where there are no first partial derivatives. It is easy to verify that if G is a FS of (7), then the third Green formula reduces equation (7) to an equivalent integral equation with boundary integrals only.

The required values of V are obtained immediately by multiplying the computed values of the solutions $u(x, y)$ of (7) by $H(x, y)$.

Unfortunately, it is very difficult to find a FS of equation (7), owing to the function f. On the other hand, it is not worthwhile to look for a FS that will surely be complicated and, consequently, difficult to handle. However, the problem may be approached in another way, i.e., by approximating equation (7) with a constant-coefficient equation.

Approximation of the transformed equation

To simplify equation (7), we note that, in practical problems, the function $a(x, y)$ does not have large gradient and curvature. So under the hypothesis that the first- and second-order partial derivatives of a are small or, in other words, that the ratio

$$R = \left|\frac{\max f}{\min f}\right| \quad \text{on } D \tag{8}$$

is close to unity, then we can replace equation (7) by the constant-coefficient equation

$$\nabla^2 \bar{u} + k^2 \bar{u} = 0 \tag{9}$$

where the constant k^2 is defined by

$$k^2 = \frac{1}{\text{meas}(D)} \int_D f(x, y)\, dD \tag{10}$$

The approximation $\bar{u}$ of the function u will simply be written as u in what follows. The average value of f can be greater than, equal to, or less than zero, but in any case equation (9) has constant coefficients and looks like the Helmholtz equation (for $k^2 = 0$ it is the Laplace equation). For any value of k^2 the corresponding FS are

$$G_1(P, Q) = -\frac{1}{4} Y_0(kr) \quad \text{for } \nabla^2 u + k^2 u = 0 \tag{11}$$

$$G_2(P, Q) = \frac{1}{2\pi} K_0(kr) \quad \text{for } \nabla^2 u - k^2 u = 0 \tag{12}$$

$$G_3(P, Q) = -\frac{1}{2\pi} \log r \quad \text{for } \nabla^2 u = 0,\ r = |PQ| \tag{13}$$

where Y_0 is the Bessel function of the second kind and zeroth order, and K_0 is the corresponding modified Bessel function. The functions G_1, G_2, and G_3 have logarithmic singularity as $r \to 0$. The numerical coefficients in formulae (11)–(13) are introduced only for the sake of a unique numerical development.

With G given by (11)–(13), the integral equation equivalent to equation (9) is

$$\frac{\alpha}{2\pi} u(P) = \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS \tag{14}$$

where α denotes the angle in P described by the vector $r = |PQ|$ when Q runs over all the boundary S. The discretization of equation (14) is carried out as usual by a BEM technique, so we do not give the detail but refer the reader to Refs.[1–4]

In the numerical examples in the following section we approximated the boundary S by linear elements, the function u by linear polynomials, and the function $\partial u/\partial n$ by a constant on each element. *The choice made for u and $\partial u/\partial n$ is suggested by the fact that linear functions have constant partial derivatives. Different approximations can be found in the literature.*

Lastly, we have to examine the boundary conditions associated with equation (2), which, in general, are of the following three types (Dirichlet, Neumann, Robin):

$$\begin{cases} V(x, y) = g_1(x, y) & \text{on } S_1 \\ \dfrac{\partial V}{\partial n} = g_2(x, y) & \text{on } S_2, S_1 \cup S_2 \cup S_3 = S \\ \dfrac{\partial V}{\partial n} + g_3(x, y)V = 0 & \text{on } S_3 \end{cases} \tag{15}$$

With the help of (5) boundary conditions (15) become

$$\begin{cases} u = \sqrt{a(x, y)}\, g_1 & \text{on } S_1 \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{2a}\dfrac{\partial a}{\partial n} u + \sqrt{a}\, g_2 & \text{on } S_2, S_1 \cup S_2 \cup S_3 = S \\ \dfrac{\partial u}{\partial n} + \left(g_3 - \dfrac{1}{2a}\dfrac{\partial a}{\partial n}\right)u = 0 & \text{on } S_3 \end{cases} \tag{16}$$

We see that the Neumann condition becomes the Robin condition, but the others remain the same.

Numerical examples

In order to assess the validity of the procedure presented in the previous sections, we solve a few problems numerically. The choice of these tests is suggested by the fact that their analytical solutions are known and a rigorous check of the results is possible.

Two percentage errors are defined to measure the error between the analytical solution V and the numerical solution $\tilde{V}$ as follows:

$$\begin{cases} e_\infty(\%) = \dfrac{\max_i |V_i - \tilde{V}_i|}{\max_D V} \cdot 100 \\ e_2(\%) = \dfrac{\sum_i (V_i - \tilde{V}_i)^2}{\max_D V} \cdot 100 \end{cases} \tag{17}$$

In (17) subscript i runs over all the points $P_i \equiv (x_i, y_i) \in D \cup S$, where a numerical value is computed. Errors (17) are strictly connected to the corresponding norms in L^∞ and L^2 and have been chosen, instead of the norms, because they are more meaningful.

Example 1. Solve the following problem:

$$\begin{cases} \nabla^2 V + \dfrac{1}{a}\nabla a \cdot \nabla V = 0 \\ \quad \text{on } D \equiv \{(x, y): 0 < x < 5; 0 < y < 5\} \\ a(x, y) = (x + 1)(y + 2) \\ V(x = 0, y) = 0 \\ V(x = 5, y) = 5 \\ \dfrac{\partial V}{\partial n} = 0 \quad \text{for } y = 0, y = 5, 0 \leqslant x \leqslant 5 \end{cases} \tag{18}$$

whose analytical solution is

$$V(x, y) = \frac{5}{\log 6}\log(1 + x) \tag{19}$$

For this problem the function f and the parameter R are

$$f(x, y) = \frac{1}{4}\left\{\frac{1}{(1 + x)^2} + \frac{1}{(2 + y)^2}\right\} \qquad R \cong 26 \tag{20}$$

and the average value of f on D is $k^2 = 0.059524$. The boundary S is discretized by 40 linear equal elements. The results were checked on the 40 boundary points plus 56 other internal points. Errors (18) were

$$e_\infty = 7.4\% \qquad e_2 = 17.9\% \tag{21}$$

The results are not bad if we consider the simplicity of the method and the fact that, in this problem, f does not have a limited variation ($R = 26$ is very far from unity). We could reduce errors (22) by discretizing differently the boundary S or, even better, by subdividing the domain D into two or three subdomains in order to cut down the ratio R in each of them. Subdividing D takes into account the behaviour of f: In this example a glance at equation (20) suggests that a suitable subdivision into subdomains D_1 and D_2 is

$$\begin{aligned} D_1 &= \{(x, y): 0 < x < 2; 0 < y < 5\} \\ D_2 &= \{(x, y): 2 < x < 5; 0 < y < 5\} \end{aligned} \tag{22}$$

In D_1 and D_2 the values of R and k^2 are, respectively,

$$\begin{aligned} R_1 &= 8.04 \qquad k_1^2 = 0.101190 \\ R_2 &= 7.49 \qquad k_2^2 = 0.031746 \end{aligned} \tag{23}$$

On the interface between D_1 and D_2 continuity of the potential u and of the flux has to be imposed.

By subdivision (22) the values of errors (18) are

$$e_\infty = 3.3\% \qquad e_2 = 5.7\% \tag{24}$$

The improvement of the results is quite apparent.

Example 2. The problem and the domain are the same as in Example 1, except that function a is defined by

$$a(x, y) = (1 + \tfrac{1}{2}x)(2 + \tfrac{1}{2}y) \tag{25}$$

The discretization of S and the test points are also the same. The function f and the parameter R are

$$f(x, y) = \frac{1}{4}\left\{\frac{1}{(2 + x)^2} + \frac{1}{(4 + y)^2}\right\} \qquad R = 9.5 \tag{26}$$

The method gives errors

$$e_\infty = 1.8\% \qquad e_2 = 1.2\% \tag{27}$$

There is good improvement in the results even though f still has a large variation. *Figure 1* shows the equipotential lines obtained from the numerical results (full lines) and a few of the analytical lines (dashed lines).

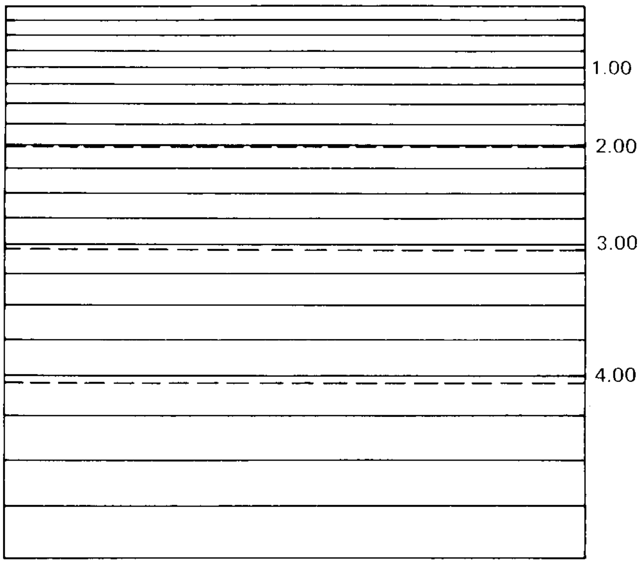

Figure 1 Equipotential lines for Example 2: (———) numerical values; (------) analytical solution

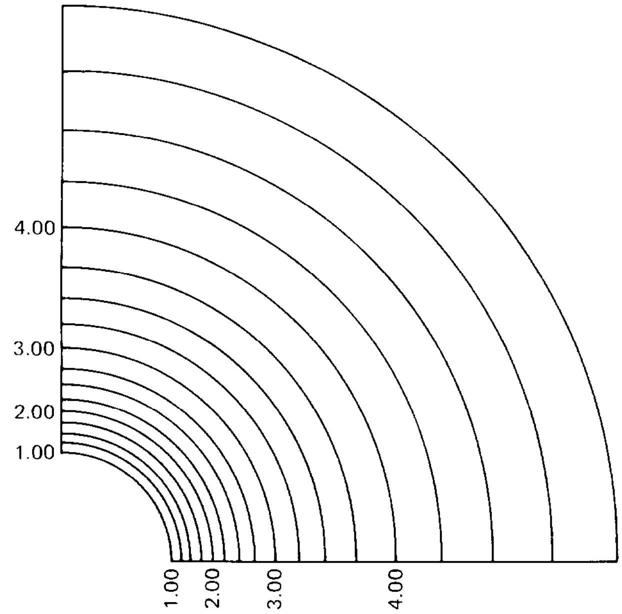

Figure 2 Equipotential lines obtained from the numerical values for Example 3

Example 3. Find the electrical potential in a cylindrical capacitor with nonhomogeneous dielectric. Mathematically the problem is defined by

$$\begin{cases} \nabla^2 V + \dfrac{1}{a}\nabla a \cdot \nabla V = 0 & \\ \quad \text{on } D \equiv \{(x, y)\colon 1 < r < 5,\ x > 0,\ y > 0\} & \\ a(x, y) = 1 + \frac{1}{2}r & r^2 = x^2 + y^2 \\ V = 1 & \text{for } r = 1 \\ V = 5 & \text{for } r = 5 \\ \dfrac{\partial V}{\partial n} = 0 & \\ \quad \text{for } y = 0,\ 1 \leqslant x \leqslant 5;\ x = 0,\ 0 \leqslant y \leqslant 5 & \end{cases} \tag{28}$$

The boundary S is discretized with 8 equal linear elements on the rectilinear sides and 10 equal elements on the curved sides (a better discretization could have been carried out if we expected larger gradients on approaching the inner boundary). For this problem f, R, and k^2 are

$$f(x, y) = \frac{1}{2}\left\{\frac{1}{2(2 + r)^2} - \frac{1}{r(2 + r)}\right\}$$
$$R = 4.92 \qquad k^2 = -0.035729 \tag{29}$$

In this case a negative value of k^2 appears, so the FS (12) must be used.

The errors obtained for this problem using 36 boundary points and 43 internal points are

$$e_\infty = 1.9\% \qquad e_2 = 6.5\% \tag{30}$$

The equipotential lines computed numerically are shown in *Figure 2*.

Conclusion

A simple procedure is presented for solving numerically the GLE. This procedure seems to give sufficiently accurate results in practical problems where physical properties have bounded variations. Some points, connected with the technique, have not been touched on in this paper for the sake of brevity, but it is the authors' intention to deal with them in a subsequent paper. In particular, it is important to estimate the error introduced by substituting equation (7) into equation (9) under certain hypotheses on the function f, and to check the influence of any error related to the evaluation of k^2. In the problems presented here, k^2 was computed analytically, but in general it has to be evaluated numerically.

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